MATH6031 Lecture 4

Recall: we have Duflo-Kontsevich isomorphisms

- For a finite-dimensional Lie algebra $\mathfrak{g}$,

$$I_{PBW} \circ J^{1/2} : H^\bullet(\mathfrak{g}, S^\bullet(\mathfrak{g})) \xrightarrow{\sim} HH^\bullet(U^\bullet(\mathfrak{g}))$$

- For a complex manifold X,

$$I_{HKR} \circ td_{T'}^{\frac{1}{2}} : H^\bullet_{\bar\partial}(X, \wedge^\bullet T') \xrightarrow{\sim} H^\bullet(\Omega^{0,\bullet}(D'^{\bullet}_{poly}), d_H + \bar\partial)$$

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§ Supermathematics

Def A super vector space (or simply superspace) is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space

$$V = V_0 \oplus V_1$$

We have a parity reversion operation

$$\Pi : (\Pi V)_0 = V_1 \text{ and } (\Pi V)_1 = V_0$$

From now on, we assume that $\dim V < +\infty$.

- Supertrace and Berezinian

Consider $X \in \mathrm{End}(V)$

$$X = \begin{pmatrix} x_{00} & x_{10} \\ x_{01} & x_{11} \end{pmatrix} \quad \text{where } x_{ij} \in \mathrm{Hom}(V_i, V_j)$$

We define the supertrace of X as

$$\mathrm{str}(X) := \mathrm{tr}(x_{00}) - \mathrm{tr}(x_{11})$$

Now suppose X is invertible. Then the Berezinian (or superdeterminant) of X is uniquely determined by the following rules

$$\begin{cases} \mathrm{Ber}(AB) = \mathrm{Ber}(A)\,\mathrm{Ber}(B) \\ \mathrm{Ber}(e^X) = e^{\mathrm{str}(X)} \end{cases}$$

If we write $X = \begin{pmatrix} X_{00} & X_{10} \\ X_{01} & X_{11} \end{pmatrix}$, then

$$\mathrm{Ber}(X) := \det(X_{00} - X_{10}X_{11}^{-1}X_{01})\det(X_{11})^{-1}$$

by the decomposition

$$\begin{pmatrix} X_{00} & X_{10} \\ X_{01} & X_{11} \end{pmatrix} = \begin{pmatrix} 1 & X_{10}X_{11}^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} X_{00} - X_{10}X_{11}^{-1}X_{01} & 0 \\ X_{01} & X_{11} \end{pmatrix}$$

- The symmetric algebra of a superspace V is defined as

$$S^{\cdot}(V) := T(V) \Big/ \left(v\otimes w - (-1)^{|v||w|} w \otimes v : v, w \text{ homog. elts in } V \right)$$

$S(V)$ admits two different $\mathbb{Z}$-gradings:

(1) symmetric degree: $\deg(v) = 1$ for $v \in V$;

deg n homogeneous piece is given by

$$S^n(V) = V^{\otimes n} / \mathfrak{S}_n$$

where $\mathfrak{S}_n$ acts on $V^{\otimes n}$ by

$$(i, i+1)\cdot(v_1 \otimes \cdots \otimes v_n) = (-1)^{|v_i||v_{i+1}|} v_1 \otimes \cdots \otimes v_{i+1} \otimes v_i \otimes \cdots \otimes v_n$$

(2) internal degree: $\deg = i \in \{0,1\}$ for elts in V_i;

deg n homogeneous piece is denoted $S(V)^n$

and we write $|x|$ for the internal degree of a homogeneous elt $x \in S(V)$.

e.g. (a) If $V = V_0$ is purely even, then $S(V) = S(V_0)$

is the usual symm. algebra of V_0. In this case,

$$S^n(V) = S^n(V_0) \quad \text{and} \quad S(V) = S(V)^0.$$

(b) If $V = V_1$ is purely odd, then $S(V) = \Lambda(V_1)$ is the usual exterior algebra of V_1. In this case,

$$S^n(V) = \Lambda^n(V_1) = S(V)^n.$$

- The (graded) exterior algebra of V is defined as

$$\Lambda^{\cdot}(V) := T(V) \Big/ \left(v \otimes w + (-1)^{|v||w|} w \otimes v : v, w \text{ homog. elts in } V \right)$$

As above, $\Lambda^{\cdot}(V)$ admits two $\mathbb{Z}$-gradings:

(1) exterior degree : $\deg(v) = 1$ for $v \in V$;

deg n homogeneous piece is given by

$$\Lambda^n(V) = V^{\otimes n} / \mathfrak{S}_n$$

where $\mathfrak{S}_n$ acts $V^{\otimes n}$ by

$$(i, i+1) \cdot (v_1 \otimes \cdots \otimes v_n) := -(-1)^{|v_i||v_{i+1}|} v_1 \otimes \cdots \otimes v_{i+1} \otimes v_i \otimes \cdots \otimes v_n.$$

(2) internal degree : $\deg = i \in \{0, 1\}$ for elts of V_{1-i};

deg n homogeneous piece is denoted by $\Lambda(V)^n$ and we write $|x|$ for the internal degree of a homogeneous elt $x \in \Lambda(V)$,

i.e. $$|v_1 \wedge \cdots \wedge v_n| = \sum_{i=1}^{n} \overset{1-|v_i|}{\cancel{\deg}(v_i)} = n - \sum_{i=1}^{n} |v_i|$$

e.g. (a) If $V = V_0$ is purely even, then $\Lambda(V) = \Lambda(V_0)$ is the usual exterior algebra of V_0, and we have

$$\Lambda^n(V) = \Lambda^n(V_0) = \Lambda(V)^n.$$

(b) If $V = V_1$ is purely odd, then $\Lambda(V) = S(V_1)$ is the usual symm. algebra of V_1, and we have
$$\Lambda^n(V) = S^n(V_1) \quad \text{and} \quad \Lambda(V) = \Lambda(V)^0$$

We have an isom. of bigraded vector spaces
$$S^\cdot(\Pi V) \xrightarrow{\sim} \Lambda^\cdot(V) \quad \text{— } (*)$$
$$v_1 \cdots v_n \longmapsto (-1)^{\sum_{j=1}^{n}(j-1)|v_j|} v_1 \wedge \cdots \wedge v_n$$

Def A graded algebra $A^\cdot$ is called graded commutative if $a \cdot b = (-1)^{|a||b|} ba \quad \forall a, b \in A^\cdot$.

e.g. • For a superspace V, $S^\cdot(V)$ is graded commutative w.r.t. the internal graded.

• For a smooth manifold M, $\Omega^\cdot(M)$ is graded commutative

$\Lambda^\cdot(V)$ is NOT graded commutative

But we can define a new product $\bullet$ on $\Lambda(V)$ by
$$v \bullet w = (-1)^{k(|w|+\ell)} v \wedge w$$
for $v \in \Lambda^k(V)$ and $w \in \Lambda^\ell(V)$

Then $(\Lambda^\cdot(V), \bullet)$ is graded commutative, and $(*)$ becomes an isom. between graded algebras.

Def A graded Lie algebra is a $\mathbb{Z}$-graded vector space $\mathfrak{g}^\cdot$ equipped with a deg 0 graded skew-symmetric linear map

i.e. $[x, y] = -(-1)^{|x||y|}[y, x]$
$$[\cdot, \cdot] : \mathfrak{g}^\cdot \times \mathfrak{g}^\cdot \to \mathfrak{g}^\cdot$$

linear map $\quad$ i.e. $[x,y] = -(-1)^{|x||y|}[y,x]$

$$[\cdot,\cdot] : \mathfrak{g}^\cdot \times \mathfrak{g}^\cdot \longrightarrow \mathfrak{g}^\cdot$$

satisfying the graded Jacobi identity:

$$[x,[y,z]] = [[x,y],z] + (-1)^{|x||y|}[y,[x,z]]$$

e.g. • Let $A^\cdot$ be a graded associative algebra.

Then A equipped with the super-commutator

$$[a,b] = ab - (-1)^{|a||b|} ba$$

is a graded Lie algebra.

• Let $A^\cdot$ be a graded associative algebra.

Then a degree k graded linear map

$$d : A \longrightarrow A$$

is called a super-derivation if

$$d(ab) = d(a)\cdot b + (-1)^{k|a|} a \cdot d(b)$$

Let $\mathrm{Der}(A) :=$ the set of all super-derivations on A

Then $\mathrm{Der}(A)$ is closed under composition inside $\mathrm{End}(A)$.

<u>Def</u> Let $\mathfrak{g}^\cdot$ be a graded Lie algebra.

(1) A graded $\mathfrak{g}$-module is a graded vector space V together with a deg 0 graded linear map

$$\mathfrak{g} \times V \longrightarrow V$$

s.t. $\quad x\cdot(y\cdot v) - (-1)^{|x||y|} y\cdot(x\cdot v) = [x,y]\cdot v$

In other words, it is a morphism

$$\mathfrak{g} \longrightarrow \mathrm{End}(V)$$

of graded Lie algebras.

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(2) If $V = A$ is a graded associative algebra, we say $\mathfrak{g}$ acts on A by derivations if the image of $\mathfrak{g} \to \mathrm{End}(A)$ lies in $\mathrm{Der}(A)$. In this case, we say A is a $\mathfrak{g}$-module algebra.

§ Hochschild cohomology (cont'd)

- HC of a graded algebra

$$A : \text{graded assoc. algebra}$$

- The (shifted) Hochschild (cochain) complex of A is defined as

$$C^{\cdot}(A,A) = \{\text{linear maps } A^{\otimes(\cdot-1)} \longrightarrow A^{\cdot}\}$$

Denote by $|\cdot|$ the degree of such a linear map. Then grading on $C^{\cdot}(A,A)$ is given by the total degree $\|\cdot\|$: for $f : A^{\otimes m} \longrightarrow A$, $\|f\| := |f| + m - 1$.

- The differential d_H is defined by

$$\big(d_H(f)\big)(a_1, \dots, a_{m+1}) = (-1)^{\|f\|(|a_1|-1)} a_1 f(a_2, \dots, a_{m+1})$$
$$+ \sum_{i=1}^{m} (-1)^{i-1+\sum_{j=1}^{i-1}|a_j|} f(a_1, \dots, a_i a_{i+1}, \dots, a_{m+1})$$
$$+ f(a_1, \dots, a_m) a_{m+1}$$

We have $d_H^2 = 0 \rightsquigarrow HH^{\cdot}(A^{\cdot})$

- $(C^{\cdot}(A,A), d_H)$ is a differential graded algebra (DGA) where the product is defined by

$$(f \cup g)(a_1, \dots, a_{m+n}) := (-1)^{|g|(|a_1| + \dots + |a_m|)} f(a_1, \dots, a_m)\, g(a_{m+1}, \dots, a_{m+n})$$

- **HC of a DGA**

For any $d \in \mathrm{Der}(A)$ and $f \in C^{\cdot}(A,A)$, define

$$\big(d(f)\big)(a_1, \dots, a_m) := d\big(f(a_1, \dots, a_m)\big) - (-1)^{\|d\|\|f\|} \sum_{i=1}^{m} (-1)^{\|d\|\left(i-1+\sum_{j=1}^{i-1} |a_j|\right)} f(a_1, \dots, d a_i, \dots, a_m)$$

i.e. $d : C^{\cdot}(A,A) \to C^{\cdot}(A,A)$ is the unique degree $|d|$ derivation for the cup product given by supercommutations on linear maps $A \to A$.

We can check that $d \circ d_H + d_H \circ d = 0$.

Therefore, if $(A^{\cdot}, d)$ is a DGA (i.e. $|d| = 1$), then $d_H + d$ is a differential on $C^{\cdot}(A,A)$ and $(C^{\cdot}(A,A), d_H + d)$ is also a DGA.

We call $HH^{\cdot}(A,d) = H^{\cdot}(C^{\cdot}(A,A), d_H + d)$ the Hochschild cohomology of the DGA $(A^{\cdot}, d)$.

Rmk $(C^{\cdot}(A,A), d_H + d)$ controls the deformations of (A,d) as an A_∞-algebra:

$HH^2(A,d) =$ infinitestimal deform$^{\text{ns}}$

$HH^3(A,d) \supset$ obstructions

Thm Let $\mathfrak{g}$ be a finite-dimensional Lie algebra.

Then $\exists$

Thm Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Then $\exists$ an isom of graded algebras

$$\underbrace{HH^\cdot(\Lambda\mathfrak{g}^*, d_c)}_{\phi} \xrightarrow{\sim} \underbrace{HH^\cdot(U(\mathfrak{g}))}_{\phi}$$

$x \in \mathfrak{g}^k$
$y \in \mathfrak{g}^l$

$\underline{x \otimes y} - \underline{y \otimes x} - \underline{[x, y]}$

$$H^\cdot(\mathfrak{g}, V) = H^\cdot(\underbrace{C^\cdot(\mathfrak{g}, V)}, d_c)$$

$$\| \quad \{\text{linear maps } \Lambda^\cdot\mathfrak{g} \to V\}$$

$$\| \quad \boxed{\Lambda\mathfrak{g}^* \otimes V}$$

$$\| \quad \Lambda\mathfrak{g}^*$$

Here we take $V = k$

V : $\mathfrak{g}$-module

Pf: Since we have $HH^\cdot(U(\mathfrak{g})) \cong \underbrace{H^\cdot(\mathfrak{g}, U(\mathfrak{g}))}$,

it suffices to show that

$$HH^\cdot(\Lambda\mathfrak{g}^*, d_c) \xrightarrow{\sim} H^\cdot(\mathfrak{g}, U(\mathfrak{g})).$$

Consider the map

$$C^\cdot(\Lambda\mathfrak{g}^*, \Lambda\mathfrak{g}^*) = \Lambda\mathfrak{g}^* \otimes T(\Lambda\mathfrak{g}) \xrightarrow{\ \ } \Lambda\mathfrak{g}^* \otimes U(\mathfrak{g}) = C^\cdot(\mathfrak{g}, U(\mathfrak{g}))$$

$\| \; \{$linear maps $(\Lambda\mathfrak{g}^*)^{\otimes m} \longrightarrow \Lambda\mathfrak{g}^*\}$

$\uparrow$ by the projection $T(\Lambda\mathfrak{g}) \twoheadrightarrow T(\mathfrak{g}) \twoheadrightarrow U(\mathfrak{g})$

which defines a morphism of DGAs.

$$(C^\cdot(\Lambda\mathfrak{g}^*, \Lambda\mathfrak{g}^*), d_H + d_c) \longrightarrow (C^\cdot(\mathfrak{g}, U(\mathfrak{g})), d_c)$$

Now we have

$$H^\cdot((\underbrace{\Lambda\mathfrak{g}^*, d_c}_{\text{DGA}}), k) \overset{(**)}{\cong} U(\mathfrak{g})$$

by a spectral sequence argument.

$H^\cdot(A, M)$

k is regarded as a $(\Lambda\mathfrak{g}^*, d_c)$-DG-bimodule via the projection $\varepsilon : \Lambda\mathfrak{g}^* \to k$

Consider the filtration on $C^\cdot(\Lambda\mathfrak{g}^*, \Lambda\mathfrak{g}^*)$ induced by $F^\cdot(\Lambda\mathfrak{g}^*)$ where $F^n(\Lambda\mathfrak{g}^*) := \bigoplus_{k\geq n} \Lambda^k\mathfrak{g}^*$.

Then $E_0^{\cdot,\cdot} = \Lambda^\cdot\mathfrak{g}^* \otimes \underbrace{C^\cdot((\Lambda\mathfrak{g}^*, d_c), k)}_{\ast}$, $d_0 = id\otimes(d_H + d_c)$

$(**) \Rightarrow \quad E_1^{\cdot,\cdot} = E_1^{\cdot,0} = \underline{\Lambda\mathfrak{g}^* \otimes U(\mathfrak{g})}$ with $\underline{d_1 = d_c}$

So the spectral sequences stabilizes at E_2 and the result follows. #

Rmk : Koszul duality for quadratic algebras.